Robust and non-robust equilibria in a strategic market game*

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Abstract

We illustrate an approach to restricting the set of equilibria in a strategic market game, based on the idea that equilibria should be robust to arbitrarily small transaction costs. Specifically, we provide a class of examples with a continuum of active Nash equilibria, of which only one survives the requirement. We also show that in a small market game a robust active equilibrium may not exist, and discuss the underlying intuition. The results somehow clarify the role of 'wash-sales' in a number of results in the literature.

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1. Introduction

In non cooperative game theory, it is standard to consider perturbations of the initial game to test if the predictions of the theory are still good predictions for "nearby" games. This type of approach has been used notably in the vast literature on refinements of the Nash equilibrium concept.[1]

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[1]For instance, perturbations underlie the essential equilibrium of [19], the perfect equilibrium of [14] and, more recently, the stability notion of [7]. See [17, 6] for thorough treatments of the literature on equilibrium refinements.

When the game represents an exchange economy with nil transaction costs, an especially relevant perturbation is to introduce slight but positive transaction costs. Indeed, an equilibrium that cannot be approximated when arbitrarily small transaction costs are allowed should be viewed as artificial.

The purpose of this paper is to illustrate the effect of this natural requirement in the strategic market game research program initiated by [15, 16, 12]. We analyze an example of a market game *à la* Postlewaite and Schmeidler [12], for which we can vary the gains of trade and the intensity of competition. For this class of exchange economies, we first show that in addition to the trivial (no-trade) equilibrium, there always exists a continuum of non-trivial Nash equilibria. This is consistent with previous studies [11]. We then consider robust equilibria, viz those Nash equilibria that can be approached by equilibria of perturbed games as transaction costs get arbitrarily small. There, we show the following. When the (potential) gains from trade are large enough compared to the degree of price manipulability, the set of robust equilibria reduces to the trivial equilibrium plus one active robust equilibrium. In particular, with corner endowments or when the number of identical agents is large enough, an active robust equilibrium exists. Besides, the corresponding allocation converges to the (unique) competitive allocation as the economy is replicated. Contrarily, when the gains from trade are not sufficient the trivial Nash equilibrium is the only robust equilibrium. The intuition for this latter result is reminiscent of the argument in Cordella and Gabszewicz [5] for why no trade can be the only Nash equilibrium in a bilateral oligopoly market game.

This shows that requiring an equilibrium to be immune to arbitrarily small transactions costs provides a powerful (and natural) approach to restricting the set of equilibria in a strategic market game. In our example, while there is a continuum of non-robust equilibria, there are at most two robust equilibria, corresponding to the Nash equilibria such that no agent trades on both sides of a given market. Further, for a 'small' market game where agents have too much power in manipulating terms of trade the trivial NE may be the only robust equilibrium.

One motivation for this paper was to contribute to the debate on what is a 'good' notion of equilibrium in the market game approach. The literature so far has focused on the issue of the activity or inactivity of trading posts (see [15, 5, 4] and also, [18, 9]). Our approach is different and complementary as our natural requirement has cutting power among *active* equilibria.

The paper is organized as follows. We present the example in section 2. Section 3 provides (standard) results for Nash equilibria of the initial market game, Γ. Section 5 analyzes the perturbed games, Γ_ε, and the robust equilibria of Γ. A final section concludes with some remarks about the effect of 'wash sales' and the comparison of the buy-and-sell with the buy-or-sell model. Some lengthy proofs are contained in an Appendix.

2. Environment

2.1. Preferences and endowments

We consider an exchange economy with two commodities $i = 1, 2$ and a finite set $\mathcal{H}$ of agents indexed by $h = 1, \ldots, 2n$ (with $n \geq 2$). Agents are divided into n type I agents and n type II agents. When no confusion should result, we use the superscript 'I'(resp. 'II') instead of h to refer to a typical type I (resp. II) agent.

Type I agents have an endowment $\left(e_1^I, e_2^I\right) = (1-\alpha, \alpha)$ and preferences represented by

$$u^I\left(x_1, x_2\right) = \ln x_1 + \beta \ln x_2, \tag{1}$$

where $\beta \geq 1$ and $0 \leq \alpha \leq \frac{1}{2}$, while type II agents have an endowment $\left(e_1^{II}, e_2^{II}\right) = (\alpha, 1-\alpha)$ and preferences given by

$$u^{II}\left(x_1, x_2\right) = \beta \ln x_1 + \ln x_2. \tag{2}$$

Hence, type I agents are relatively endowed in good 1 and would prefer to consume more good 2, while type II agents are relatively more endowed in good 2 and would prefer to consume more good 1.

For this economy, there is a (unique) competitive equilibrium, which delivers the competitive allocation

$$\left(\bar{x}_1^I, \bar{x}_2^I\right) = \left(\frac{1}{1+\beta}, \frac{\beta}{1+\beta}\right), \tag{3}$$

$$\left(\bar{x}_1^{II}, \bar{x}_2^{II}\right) = \left(\frac{\beta}{1+\beta}, \frac{1}{1+\beta}\right). \tag{4}$$

The (potential) gains from trade increase as β increases and as α decreases.

2.2. The market games, Γ and Γ_ε

Trade is organized along the line of the inside money market game [12].

There are two markets (or, trading posts), one for each commodity. On market i, all agents submit a non negative offer of commodity, q_i^h, and a non negative bid in term of inside money, b_i^h. The price on market i is formed according to the standard Shapley-Shubik rule:

$$p_i = \begin{cases} B_i/Q_i & \text{if } Q_i \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $B_i = \sum\limits_{\mathcal{H}} b_i^h$ and $Q_i = \sum\limits_{\mathcal{H}} q_i^h$ denote the aggregate bid and offer on that market. Subsequently we use the convention $1/p_i = 0$ whenever $p_i = 0$.

Agents are free to buy and sell on a same post. They can issue inside money at no cost, but must be able to cover their bids with the (monetary) gains from their sales. Specifically, it is postulated that an agent that goes bankrupt has all his bids and offers confiscated (see [11] for a discussion).

Following Rogawski and Shubik [13], we model transaction costs as consuming part of the commodities offered in transaction. We further restrict ourselves to the following linear and homogeneous specification: when an agent offers q_i^h on market i, an additional quantity εq_i^h ($\varepsilon \geq 0$) is needed in order to place that offer.[2]

Let S^h denote the strategy set of agent h, and $S = S^1 \times \ldots \times S^{2n}$ the set of strategy profiles, with generic element $\sigma = (\sigma_h)_{h \in \mathcal{H}}$. To single out the strategy of a given agent h, we will sometimes write the strategy profile as $\left(\sigma^h, \sigma^{-h}\right)$. For a given $h \in \mathcal{H}$, let

$$B_i^{-h} = \sum_{\mathcal{H} \setminus \{h\}} b_i^{h'} \qquad \text{and} \qquad Q_i^{-h} = \sum_{\mathcal{H} \setminus \{h\}} q_i^{h'}.$$

With these specifications, agent h chooses his strategy σ^h in the set

$$S^h = \left\{ \left(q_1^h, b_1^h, q_2^h, b_2^h\right) \in I\!R_+^4 \quad | \quad (1+\varepsilon)\, q_i^h \leq e_i^h \quad \forall i = 1,2 \right\}. \tag{6}$$

and, given others' strategies, does not go bankrupt whenever

$$b_1^h + b_2^h \leq q_1^h p_1 + q_2^h p_2. \tag{7}$$

Given a strategy profile $\sigma = \left(\sigma^h\right)_{h \in \mathcal{H}} \in S$, final allocations are determined by the mapping:

$$x_i^h(\sigma) = \begin{cases} e_i^h - (1+\varepsilon)\, q_i^h + \frac{b_i^h}{p_i} & \text{if (7) holds,} \\ e_i^h - (1+\varepsilon)\, q_i^h & \text{otherwise.} \end{cases} \qquad \forall i = 1,2, \tag{8}$$

It easily follows from (8) that (7) holds with equality at the optimum for h.

This completes the description of the market game. A Nash equilibrium is simply a strategy profile $\sigma \in S$ such that any agent maximizes $u^h\left(\sigma^h, \sigma^{-h}\right)$ over S_h. (With a slight abuse of notation, we write $u^h(\sigma)$ for $u^h\left(x_1^h(\sigma), x_2^h(\sigma)\right)$).

When $\varepsilon = 0$, our game reduces to an example of the canonical market game studied in Peck et al. [11]. Accordingly, in what follows we will sometimes refer to this case as the "initial game", Γ, and to the games with transaction costs as the "perturbed games", $(\Gamma_\varepsilon)_{\varepsilon > 0}$.

[2] This is without loss in generality. In particular, the assumption that transaction costs depend on offers q_i^h but not on bids b_i^h is a simplification. All the analysis goes through if we assume general costs functions $\varepsilon \cdot c_i\left(q_i^h, b_i^h\right)$ provided that $c_i(.,.)$ is weakly increasing in both arguments and strictly increasing in one argument.

3. Nash equilibria of Γ

This section is concerned with Nash equilibria (NE) of the original game Γ. Note that in the case $\alpha > 0$ our economy satisfies all the assumptions in Peck et al. [11], so that existence and indeterminacy of *interior* equilibria can be deduced from their general analysis.

In the game Γ, agent h chooses his strategy $\sigma^h \in S^h$ to maximize $u^h(\sigma_h, \sigma_{-h})$ subject to (7). A key feature of this framework is that agents have a continuum of best reply strategies. More precisely, two strategies that differ by 'wash sales' trades yield the same utility. Formally:

Lemma 1. *Let* $\left(Q_1^{-h}, B_1^{-h}, Q_2^{-h}, B_2^{-h}\right) > 0$. *Consider a feasible strategy* σ^h,[3] *and any* $\tilde{\sigma}_h \in S^h$. *The following assertion are equivalent: (i)* $\tilde{\sigma}_h$ *induces the same allocation (as* σ^h*); (ii)* $\forall i = 1, 2$ *either* $\Delta q_i = \Delta b_i = 0$ *or*

$$\frac{\Delta b_i}{\Delta q_i} = \frac{b_i^h + B_i^{-h}}{q_i^h + Q_i^{-h}} = \frac{b_i^h + \Delta b_i + B_i^{-h}}{q_i^h + \Delta q_i + Q_i^{-h}}, \tag{9}$$

where $\Delta b_i \stackrel{def}{=} \tilde{b}_i^h - b_i^h$ *and* $\Delta q_i \stackrel{def}{=} \tilde{q}_i^h - q_i^h$ *denote the changes in bids and offers on post* i. *In particular,* $\tilde{\sigma}_h$ *induces the same prices and leaves* h'*s "budgetary position" unaltered.*

Proof. Using (8) with $\varepsilon = 0$, (i) is equivalent to

$$\Delta q_i = \left(b_i^h + \Delta b_i\right) \frac{q_i^I + \Delta q_i + Q_1^{-h}}{b_i + \Delta b_i + B_1^{-h}} - b_i^h \frac{q_i^h + Q_i^{-h}}{b_i^h + B_i^{-h}}, \qquad \forall i = 1, 2. \tag{10}$$

It can be easily verified that (10) amounts to $\Delta b_i \left(q_i^h + Q_i^{-h}\right) = \Delta q_i \left(b_i^h + B_i^{-h}\right)$, which implies $\Delta b_i = 0$ whenever $\Delta q_i = 0$, and is equivalent to the first equality in (9) when $\Delta q_i \neq 0$. The second equality is implied by the first. $\square$

When two strategies satisfy the above conditions, we say that they are 'equivalent'. Note that lemma 1 implies that any strategy is equivalent to a 'maximal' strategy in which h offers all his endowment. This allows for a straightforward characterization of the set of best reply strategies $\mathcal{BR}\left(\sigma^{-h}\right)$:[4]

Proposition 1. *Let* h *be a type I agent. Given strictly positive of other agents' aggregate bids and offers,* $\left(Q_1^{-h}, B_1^{-h}, Q_2^{-h}, B_2^{-h}\right)$, *a strategy* $\sigma^h \in S^h$

[3] By 'feasible', we mean that $\sigma_h \in S_h$ and satisfies the budget constraint (7).

[4] Proposition 1 can also be obtained by a direct application of proposition 2.4 in [11]

is optimal for h iff the budget constraint (7) holds with equality and the strategy satisfies the first order condition

$$\frac{\alpha - q_2^h + b_2^h \frac{Q_2}{B_2}}{1 - \alpha - q_1^h + b_1^h \frac{Q_1}{B_1}} \frac{1}{\beta} = \frac{Q_1^{-h}}{B_1^{-h}} \left(\frac{B_1}{Q_1}\right)^2 \frac{B_2^{-h}}{Q_2^{-h}} \left(\frac{Q_2}{B_2}\right)^2. \tag{11}$$

Proof. The optimum can always be attained by a 'maximal' strategy with $q_i^h = e_i^h$. For such strategies, h solves

$$\begin{aligned} &\max_{(b_1^h, b_2^h) \geq 0} \ \ln \tfrac{b_1^h}{b_1^h + B_1^{-h}} Q_1 + \beta \ln \tfrac{b_2^h}{b_2^h + B_2^{-h}} Q_2 \\ &\text{s.t. } b_1^h + b_2^h \leq \tfrac{1-\alpha}{Q_1} \left(b_1^h + B_1^{-h}\right) + \tfrac{\alpha}{Q_2} \left(b_2^h + B_2^{-h}\right). \end{aligned} \tag{$\mathcal{P}_0$}$$

$(\mathcal{P}_0)$ is a convex maximization problem with a unique solution. Clearly this solution is interior, and is given by the unique $\left(\bar{b}_1^h, \bar{b}_2^h\right)$ that satisfies the budget constraint with equality and the first order condition

$$\frac{b_2^h \frac{Q_2}{B_2}}{b_1^h \frac{Q_1}{B_1}} \frac{1}{\beta} = \frac{Q_1^{-h}}{B_1^{-h}} \left(\frac{b_1^h + B_1^{-h}}{Q_1}\right)^2 \frac{B_2^{-h}}{Q_2^{-h}} \left(\frac{Q_2}{b_2^h + B_2^{-h}}\right)^2. \tag{12}$$

Let $\bar{\sigma}^h = \left(1 - \alpha, \bar{b}_1^h, \alpha, \bar{b}_2^h\right)$. Note that $\bar{\sigma}^h$ is the unique *maximal* optimal strategy or, equivalently, the unique strategy with $q_i^h = e_i^h$ satisfying (7) and (11). Now, any strategy in $\mathcal{BR}\left(\sigma^{-h}\right)$ is equivalent to an optimal strategy with $q_i^h = e_i^h$, which by unicity is $\bar{\sigma}^h$. Given that (11) only depends on the prices and allocation attained at σ^h, strategies equivalent to $\bar{\sigma}^h$ also satisfy (7) and (11). Conversely, let σ^h satisfy (7) and (11). Then σ^h is equivalent to a maximal strategy that satisfies (7) and (11), that is to $\bar{\sigma}^h$. Hence, $u^h\left(\sigma^h, \sigma^{-h}\right) = u^h\left(\bar{\sigma}^h, \sigma^{-h}\right)$, implying that $\sigma^h \in \mathcal{BR}\left(\sigma^{-h}\right)$. □

An analogous characterization holds for type II agents.

To illustrate existence and indeterminacy, we restrict attention to Completely Symmetric Nash Equilibria (CSNE) in which all type I agents play $\sigma^I = (q, b, q', b')$ and, symmetrically all type II agents play $\sigma^{II} = (q', b', q, b)$. For such situations, budgetary constraints and first order conditions for both types are identical. Using proposition 1, an admissible collection $(q, q', b, b') \in [0, 1-\alpha] \times [0, \alpha] \times I\!R_+^2$ is a CSNE if and only if it satisfies the following system

$$b + b' = \frac{q}{n(q + q')} n\left(b + b'\right) + \frac{q'}{n(q + q')} n\left(b + b'\right), \tag{13}$$

$$\frac{\alpha - q' + b' \frac{q+q'}{b+b'}}{1 - \alpha - q + b \frac{q+q'}{b+b'}} \frac{1}{\beta} = \frac{(n-1)q + nq'}{(n-1)b + nb'} \frac{nb + (n-1)b'}{nq + (n-1)q'}. \tag{14}$$

The budget constraint (7) is always satisfied (this reflects money conservation) Thus we have 1 equation to pin down 4 variables. The model being one of *inside* money, one degree of freedom reflects pure nominal indeterminacy.[5] Accordingly, from now on we work with 'normalized' bids. Given that $b + b' > 0$ necessarily holds at a CSNE with trade, we normalize bids by $b + b' = 1$.

Thus, an active CSNE of Γ is a collection $(q, q', b, 1-b) \in [0, 1-\alpha] \times [0, \alpha] \times [0, 1]^2$ such that

$$\frac{\alpha + q - (q + q')\, b}{1 - \alpha - q + (q + q')\, b} \frac{1}{\beta} = \frac{(n-1)\, q + nq'}{nq + (n-1)\, q'} \frac{n - 1 + b}{n - b}. \tag{15}$$

Inspection of equation (15) shows that for a given vector of offers $(q, q') \in [0, 1-\alpha] \times [0, \alpha]$, there is a unique root $b \in I\!R$. To complete the description of the set of active CSNE, we simply need to determine which offer strategies are compatible with an equilibrium. This yields:

Proposition 2. *Let $b^* (q, q')$ denote the unique solution to (15). The set of* active *CSNE of* Γ *is* $\left\{ (q, q', b, 1-b) \mid b = b^* (q, q'),\ (q, q') \in \mathbf{Q} \right\}$, *where* $\mathbf{Q} \subset]0, 1-\alpha] \times [0, \alpha]$ *is the subset of offers* (q, q') *such that*

$$\left(\frac{\alpha + q}{1 - \alpha - q} - \beta \right) q' \geq q \frac{n}{n-1} \left(\beta \left(\frac{n-1}{n} \right)^2 - \frac{\alpha + q}{1 - \alpha - q} \right). \tag{16}$$

In particular, an active NE exists.

Proof. See appendix A. □

Hence, the game Γ always possesses a continuum of symmetric equilibria, the dimension of which is 2 when $\alpha > 0$ and 1 otherwise. For our purposes, one noteworthy consequence of proposition 2 is that the set of *active* CSNE has a unique element with maximal trade corresponding to the Sell-All model (viz, $(q, q') = (1-\alpha, \alpha)$) and, more importantly, a unique limit point with minimal trade,

$$\hat{\mathbf{q}} \stackrel{def}{=} \left\{ (\hat{q}, \hat{q}') \in \bar{\mathbf{Q}} \mid \forall\, (q, q') \in \mathbf{Q},\ q \geq \hat{q} \text{ and } q' \geq \hat{q}' \right\}, \tag{17}$$

where "more trade" means that agents put more goods to the market.[6,7] We shall comment on $\hat{\mathbf{q}}$ in the next section when studying robust equilibria.

[5] Given any solution (q, q', b, b') to (14), (q, q', mb, mb') is also a solution $\forall m \in I\!R_+$.

[6] In (17), $\bar{\mathbf{Q}}$ denotes the closure of $\mathbf{Q}$.

[7] Formally, one candidate CSNE A involves "more trade" than candidate equilibrium B iff $q_A \geq q_B$ and $q'_A \geq q'_B$. Clearly, "more goods to the market" only provides a partial order on the set of (symmetric) equilibria.

As the next result shows, the indeterminacy in strategies space comes with real indeterminacy. There is a continuum of (symmetric) equilibrium allocations, and CSNE in which agents put more goods to the market are associated with allocations closer to the competitive outcome.[8] Precisely,

Proposition 3. *There exists a continuous, strictly increasing function χ : $\mathbf{Q} \rightarrow I\!R_+$ such that the allocation attained at an* active *CSNE writes*

$$\left(x_1^I, x_2^I\right) = \left(1 - \alpha - \chi\left(q, q'\right), \alpha + \chi\left(q, q'\right)\right) = \left(x_2^{II}, x_1^{II}\right). \tag{18}$$

The set of symmetric equilibrium allocations is therefore given by

$$\left\{\mathbf{x}_0\right\} \cup \left\{\lambda \hat{\mathbf{x}} + (1 - \lambda)\check{\mathbf{x}},\ \lambda \in [0, 1]\right\}, \tag{19}$$

where $\mathbf{x}_0 = (1 - \alpha, \alpha)$ is the autarkic allocation, and $\hat{\mathbf{x}}$, $\check{\mathbf{x}}$ the allocation attained at the active CSNE with minimal and maximal trade, respectively.

Proof. See appendix B □

This terminates the characterization of symmetric NE for the case in which there are no transaction costs. Note that propositions 2 and 3 are consistent with the structural results of Peck et al. [11] regarding the dimension of the set of NE in strategy space and allocation space.[9]

4. Robust equilibria

We now ask which NE of the initial game can be approached by equilibria of the perturbed games with strictly positive transaction costs as transaction costs vanish. We say that such equilibria are robust.

Formally,

Definition 1. *A Nash equilibrium σ of the game Γ is "robust" if there exists a sequence $\{^n\varepsilon, ^n\sigma\}_{n=1}^{\infty}$ where $^n\varepsilon \in I\!R_+$ and $^n\sigma\ (\in S)$ is a NE of the perturbed games $\Gamma_{^n\varepsilon}$ such that* $\lim_{n\to\infty} {}^n\varepsilon = 0$ *and* $\lim_{n\to\infty} {}^n\sigma = \sigma$.

[8]This "comparative statics" result among CSNE is consistent with the effect of short sales and market thickness in mitigating the impact of imperfect competition in [10].

[9]Peck et al. [11] show that, generically, the set of NE is a manifold of dimension $L \cdot N$ in normalized strategy space and $L \cdot (N - 1)$ in allocation space, where L is the number of goods, and N the number of players. In our setting, $L = 2$, and $N = 2n$. Restricting to CSNE, the formulae give dimension 2 and 1, respectively.

4.1. NE of perturbed game Γ_ε

In this section, we analyse equilibria of the perturbed games Γ_ε, $\varepsilon > 0$.

We first start with one intuitive and crucial result stating that—in sharp contrast with the previous case—it is never a best reply to buy and sell on the same trading post when any transaction costs are involved.

Lemma 2. *Let $\varepsilon > 0$. Any best reply in Γ_ε satisfies $b_i^h \cdot q_i^h = 0$.*

Proof. Assume the contrary, viz $b_i^h > 0$ and $q_i^h > 0$ for a candidate best reply $\tilde{\sigma}^h$. We construct a profitable deviation $\tilde{\sigma}^h$ by subtracting a small amount of wash-sales (conveniently defined) on post i. Formally, for $\eta > 0$, set $\tilde{b}_i^h = b_i^h - \eta$ and $\tilde{q}_i^h$ such that

$$\tilde{q}_i^h - q_i^h = \frac{q_i^h + Q_i^{-h}}{b_i^h + B_i^{-h}} \left(\tilde{b}_i^h - b_i^h\right). \tag{20}$$

As $b_h^{i,s} > 0$ and $q_h^{i,s} > 0$, we can choose $\eta > 0$ such that $\tilde{b}_i^h > 0$ and $\tilde{q}_i^h > 0$, so that $\tilde{\sigma}^h \in S^h$. Now, by lemma 1, $\tilde{\sigma}^h$ satisfies condition (7) and delivers the allocation $x_i^h\left(\tilde{\sigma}_h, \sigma_{-h}\right) = x_i^h\left(\sigma_h, \sigma_{-h}\right) + \varepsilon\left(q_i^h - \tilde{q}_i^h\right) > x_i^h\left(\sigma_h, \sigma_{-h}\right)$ and $x_j^h\left(\tilde{\sigma}_h, \sigma_{-h}\right) = x_j^h\left(\sigma_h, \sigma_{-h}\right)$ for $j \neq i$. The contradiction follows. $\square$

Now, using the highly symmetric nature of our example, we can show that equilibria of Γ_ε are necessarily completely symmetric—a property that *does not* hold for Γ. Further, we can rule out counterintuive situations in which agents end up buying their less preferred good.

Proposition 4. *Let $\varepsilon > 0$. Any NE of Γ_ε is a CSNE. Besides, type I agents do not bid on market 1 and do not sell on market 2. Symmetrically, type II agents do not bid on market 2 and do not sell on market 1.*

Proof. See appendix D. $\square$

We are now in a position to fully characterize active equilibria of Γ_ε. First note that the trivial equilibrium ($\sigma = 0$) is always a NE, $\forall\varepsilon$. Given proposition 4, at a candidate NE with trade a type I agent chooses $\left(q_1^I, b_2^I\right)$ to solve

$$\max_{\left(q_1^h, b_2^h\right) \in \left[0, \frac{1-\alpha}{1+\varepsilon}\right] \times I\!R_+} \ln\left(1 - \alpha - (1+\varepsilon)\, q_1^I\right) + \beta \ln\left(\frac{b_2^I}{b_2^I + B_2^{-I}} Q_2 + \alpha\right), \tag{21}$$

subject to the budget constraint

$$b_2^I \leq \frac{q_1^I}{q_1^I + Q_1^{-I}} B_1. \tag{22}$$

As we show in appendix C, this problem admits a unique solution. The unique best response strategy is $\left(q_1^I, b_2^I\right) = (0,0)$ if other players' strategies are such that

$$\beta \frac{Q_1^{-I}}{Q_2} \frac{B_2^{-I}}{B_1} \leq \frac{\alpha}{1-\alpha} (1+\varepsilon) . \tag{23}$$

When (23) does not hold, the unique best response strategy is interior, $\left(q_1^I, b_2^I\right) > 0$, and satisfies (22) with equality and the first order condition

$$-\frac{1+\varepsilon}{1-\alpha-(1+\varepsilon) q_1^I} + \beta B_1 Q_2 \frac{Q_1^{-I}}{(Q_1)^2} \frac{B_2^{-I}}{(B_2)^2} \left(\frac{b_2^I}{B_2} Q_2 + \alpha\right)^{-1} = 0. \tag{24}$$

Now in equilibrium, type I and type II agents play the symmetric strategies $\left(q_1^I, b_2^I\right) = \left(q_2^{II}, b_1^{II}\right) \stackrel{def}{=} (q, b)$. From (24), at an equilibrium with trade, $q > 0$ satisfies

$$(1+\varepsilon)(q+\alpha) = \beta \left(\frac{n-1}{n}\right)^2 (1-\alpha-(1+\varepsilon) q) , \tag{25}$$

yielding

$$q_\varepsilon^* = \frac{\beta \left(\frac{n-1}{n}\right)^2 (1-\alpha) - (1+\varepsilon)\alpha}{\left(\beta \left(\frac{n-1}{n}\right)^2 + 1\right)(1+\varepsilon)}. \tag{26}$$

An active equilibrium exists—and is unique—if and only if $q_\varepsilon^* > 0$, that is

$$\beta \frac{1-\alpha}{\alpha} > \left(\frac{n}{n-1}\right)^2 (1+\varepsilon) . \tag{27}$$

The next proposition summarizes the discussion.

Proposition 5. *Let $\varepsilon > 0$. If condition (27) does not hold, the trivial equilibrium (autarky) is the only Nash equilibrium of Γ_ε. If (27) holds, there are two equilibria: autarky, and a unique*[10] *active equilibrium in which type I agents play $\left(q_1^I, b_2^I\right) = (q_\varepsilon^*, b)$ and type II agents play $\left(q_2^{II}, b_1^{II}\right) = (q_\varepsilon^*, b)$.*

The allocation attained at the active equilibrium can be easily computed as

$$\left(x_1^I, x_2^I\right) = \left(1-\alpha-(1+\varepsilon) q_\varepsilon^*, \alpha + q_\varepsilon^*\right) = \left(x_2^{II}, x_1^{II}\right) . \tag{28}$$

Note also that, consistent with the intuition, higher transaction costs make the existence of an active equilibrium less likely.

[10] Unicity refer to *normalized* bids.

4.2. Taking the limit

We now characterize the set of robust equilibria of Γ.

First note that the trivial equilibrium is (trivially) robust, as $\sigma = 0$ is a Nash equilibrium of Γ_ε for all $\varepsilon \geq 0$. Now, inspection of (27) reveals that when

$$\beta \frac{1-\alpha}{\alpha} > \left(\frac{n}{n-1}\right)^2, \tag{29}$$

the perturbed games Γ_ε admit a unique active NE for ε sufficiently small, while no Γ_ε with $\varepsilon > 0$ has a non trivial NE when the converse hold. From proposition 5, we get

Proposition 6. *Autarky is the only robust equilibrium if condition (29) does not hold. If (29) holds, there are 2 robust equilibria: autarky, and a unique robust* active *equilibrium in which*

$$q_1^I = q_2^{II} = q^* \stackrel{def}{=} \frac{\beta \left(\frac{n-1}{n}\right)^2 (1-\alpha) - \alpha}{\beta \left(\frac{n-1}{n}\right)^2 + 1} \tag{30}$$

Proof. Follows from a straightforward continuity argument. □

The above result illustrates two distinct phenomena. First, it shows that our robustness requirement provides a powerful approach to restricting the set of Nash equilibria. Indeed—when (29) holds—Γ admits a great multiplicity of active Nash equilibria but only one active *robust* equilibrium, which corresponds to the unique NE in which no agent simultaneously buys and sells on a same trading post.[11] The intuition is as follows. In the initial game agents only care about their net trades, while they also care about their gross trades in the perturbed games. As $\varepsilon \to 0$, this leads to the selection—among the many best replies of the initial game—of the unique best response strategy minimizing gross trades, that is that without wash-sales. When condition (29) is satisfied, this is consistent with an active equilibrium. In particular, for the special case of a large economy ($n \to \infty$) or for corner endowments ($\alpha = 0$), an *active* robust equilibrium exists.[12] As one can easily verify, the allocation attained at the robust equilibrium converges to the competitive allocation as the economy is replicated.

Secondly, proposition 6 shows that a non trivial equilibrium may fail to exist. In our example, autarky is the only robust equilibrium if

$$\beta \frac{1-\alpha}{\alpha} \leq \left(\frac{n}{n-1}\right)^2. \tag{31}$$

[11] Uniqueness is due to our specific example (which also has a unique competitive allocation). The general property is *local* uniqueness [2].

[12] Recall that $\beta \geq 1$ and $\frac{1}{2} \geq \alpha \geq 0$, implying that (29) holds for large enough n.

We now argue that this situation arises for ‘good’ reasons, intimately related to the strategic market game approach. For the economy under study, the left hand side of (31) is a measure of the gains from trade,[13] while the right hand side relates to the degree of “price manipulability”. Condition (31) then states that a robust active equilibrium may not exist in a small market game when agents have too much power in manipulating prices. Quite interestingly, this intuition is analogous to that in Cordella and Gabszewicz [5] explaining why autarky can be the only Nash equilibrium in a small market game with linear preferences.

We conclude this section with one observation on the set of Nash equilibria of the initial game. The analysis conducted in section 3 shows that Γ always possesses a continuum of *active* equilibria. However, the condition identified above corresponds with a sort of ‘discontinuity’ in the set of equilibria. When condition (29) holds, there is strictly positive trade at the

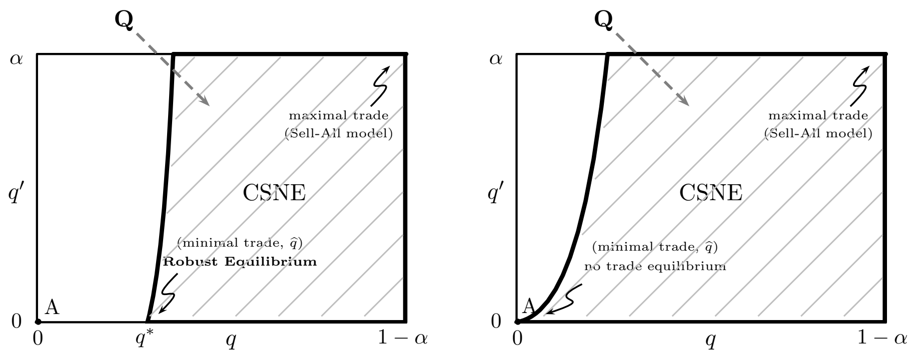

Figure 1: Set of CSNE of game Γ. *Left*: (29) holds, $\alpha = 0.3, \beta = 1.5, n = 6$. *Right*: (29) does not hold, $\alpha = 0.3, \beta = 1.5, n = 2$.

(limit) equilibrium with minimal trade defined in (17) and the set of active CSNE is separated from the trivial equilibrium. Further, the active equilibrium with minimal trade corresponds to our robust active equilibrium, $\hat{q} = (q^*, 0)$. This is illustrated in the left panel of figure 1. Contrarily, when the converse condition (31) holds, $\hat{q}$ corresponds to the no trade equilibrium. In that case there are active equilibria arbitrarily close to the trivial NE (see the right panel of figure 1).

[13]Precisely, the LHS is the marginal rate of substitution between both goods, taken at the endowment point. In our highly structured example, this is sufficient to measure the (potential) gains from trade.

5. Some tentative conclusion

In our view, good equilibria in strategic market games should be robust to arbitrarily small transaction costs. Using a class of examples, we show that this provides a powerful approach to restricting the set of equilibria in the canonical (Buy-and-Sell) market game.[14] By eliminating 'wash sales' our robustness requirement selects out equilibria in which no agent simultaneously buys and sells on a same trading post. In that sense, our approach provides an argument in favor of the Buy-or-Sell specification of the strategic market game.

We also provide examples of small, well behaved market games in which a non trivial robust equilibrium fails to exist, because agents have too much power in manipulating prices. For the same primitives, other, non trivial equilibria may exist as 'wash sales' drive down agents' ability to influence prices thereby relaxing the conditions required for existence. However, those are *fragile* equilibria that vanish with the slightest transaction costs. Incidentally, this shows that the assumption of *strictly* nil transaction costs is crucial to the existence result of [11].

Appendix

A. Proof of proposition 2.

Finding $(q, q', b) \in [0, 1-\alpha] \times [0, \alpha] \times [0, 1]$ that satisfy (15) amounts to finding solutions to $f(q, q', b) = 0$, where

$$f\left(q, q', b\right) = \frac{\alpha + q - (q + q')\, b}{1 - \alpha - q + (q + q')\, b} - \beta \frac{(n-1)\, q + nq'}{nq + (n-1)\, q'} \frac{n - 1 + b}{n - b}. \qquad (32)$$

The function $f(q, q', .)$ is continuous and strictly decreasing on the relevant range, $b \in [0, 1]$. A necessary and sufficient condition for existence of a root $b^* \in [0, 1]$ is therefore that $f(q, q', 0) \geq 0$ and $f(q, q', 1) \leq 0$. Using (32), this yield the two conditions

$$\frac{\alpha + q}{1 - \alpha - q} \geq \beta \frac{(n-1)\, q + nq'}{nq + (n-1)\, q'} \frac{n-1}{n}, \qquad (33)$$

$$\frac{\alpha - q'}{1 - \alpha - q'} \leq \beta \frac{(n-1)\, q + nq'}{nq + (n-1)\, q'} \frac{n}{n-1}. \qquad (34)$$

[14] The analysis for *generic* market games is conducted in a companion paper, [2]. In [3], we apply our robustness requirement to the multiple trading post variant developed by [8, 1]. There, we show that equilibria with price dispersion are *not* robust.

We first show that the latter condition is satisfied $\forall (q, q') \in [0, 1-\alpha] \times [0, \alpha]$. With a few manipulation, (34) is equivalent to

$$q' \left(\frac{\alpha - q'}{1 - \alpha - q'} - \beta \left(\frac{n}{n-1} \right)^2 \right) \frac{n-1}{n} \leq q \left(\beta - \frac{\alpha - q'}{1 - \alpha - q'} \right). \tag{35}$$

Observe that $\beta > \frac{\alpha}{1-\alpha}$ by our assumptions on parameters. Hence, we have for $q' \geq 0$

$$\frac{\alpha - q'}{1 - \alpha - q'} < \beta < \beta \left(\frac{n}{n-1} \right)^2, \tag{36}$$

from where it follows that the LHS of (35) is strictly negative, and the RHS strictly positive over the relevant range. Hence, (34) is always satisfied, and we need only consider (33). With a few manipulation, this necessary and sufficient condition can be rewritten equivalently as

$$\left(\frac{\alpha + q}{1 - \alpha - q} - \beta \right) q' \geq q \frac{n}{n-1} \left(\beta \left(\frac{n-1}{n} \right)^2 - \frac{\alpha + q}{1 - \alpha - q} \right), \tag{37}$$

that is the condition in proposition 2. Observe that for $q = 0$, no $q' > 0$ is compatible with equilibrium (as $\beta > \frac{\alpha}{1-\alpha}$). The special case $(q, q') = (1-\alpha, \alpha)$ implies that an active NE exists. This terminates the proof.

We now provide further characterization of the set $\mathbf{Q}$ of pairs $(q, q') \in [0, 1-\alpha] \times [0, \alpha]$ that satisfy (37). First note that the function $\phi : q \mapsto \frac{\alpha+q}{1-\alpha-q}$ is continuous and increasing over $[0, 1-\alpha[$, with $\phi([0, 1-\alpha[) = [\frac{\alpha}{1-\alpha}, +\infty[$. There are three relevant cases. *Case 1.* $\phi(q) \geq \beta$. The LHS of (37) is positive, and the RHS is negative, so that (37) holds for any relevant q'. *Case 2.* $\beta > \phi(q) > \beta \left(\frac{n-1}{n}\right)^2$. The LHS and the RHS are negative, and (37) can be rewritten as

$$q' \leq F(q) \stackrel{def}{=} q \frac{n}{n-1} \frac{\phi(q) - \beta \left(\frac{n-1}{n}\right)^2}{\beta - \phi(q)}. \tag{38}$$

Note that $F(q) = 0$ for $q = 0$ and $\phi(q) = \beta \left(\frac{n-1}{n}\right)^2$, that is $q = q^*$ defined in (30), and that $\lim_{\phi^{-1}(\beta)} F = +\infty$. Further, $F(q)$ is increasing and continuous whenever $\beta > \phi(q) > \beta \left(\frac{n-1}{n}\right)^2$ because it can be expressed as the composition of continuous increasing function, $F(q) = G(\phi(q))$, where $G : x \mapsto x \frac{x - \beta \left(\frac{n-1}{n}\right)^2}{\beta - x}$. *Case 3.* $\beta \left(\frac{n-1}{n}\right)^2 \geq \phi(q)$. The LHS is negative, and the RHS positive, so that (37) cannot hold for any $(q, q') \neq (0, 0)$ in this region. Note that case 2 arises only if $\phi(0) < \beta \left(\frac{n-1}{n}\right)^2$, that is if condition (29) in section 4 holds. This completes the characterization, and shows that $\hat{q}$ defined in (17) is unique. See figure 1. $\square$

B. Proof of proposition B

Let $\left(x_1^I, x_2^I\right) = (x, x') = \left(x_2^{II}, x_1^{II}\right)$ denote the allocation attained at a CSNE. Clearly, $x' = 1 - x$. Further, we observe that $x' \geq \alpha$ by individual rationality (agents can always attain their endowment utility level by playing $\sigma^h \equiv 0$) and that $x' < 1$ because of the boundary properties of (1) and (2). For a symmetric equilibrium, and normalized bids ($b + b' = 1$), the allocation rule (8) gives

$$x = 1 - \alpha - q + b\left(q + q'\right) = 1 - \alpha - \chi, \tag{39}$$

$$x' = \alpha - q' + (1 - b)\left(q + q'\right) = \alpha + \chi, \tag{40}$$

where $\chi = q - (q + q')\, b$. Using $\frac{\chi - q}{q + q'} = b$, the FOC (15) can be expressed as

$$\frac{\alpha + \chi}{1 - \alpha - \chi} = \beta \frac{(n-1)\, q + nq'}{nq + (n-1)\, q'} \frac{(n-1)\,(q + q') + \chi - q}{n\,(q + q') + q - \chi}, \tag{41}$$

or, equivalently,

$$\frac{\alpha + \chi}{1 - \alpha - \chi} = \beta \left(1 - \frac{\chi}{(n-1)\, q + nq' + \chi}\right)\left(1 - \frac{\chi}{nq + (n-1)\, q'}\right) \tag{42}$$

Let $F(\chi, q, q')$ denote the difference between the LHS and the RHS of (42). From (40) and $x' \in [\alpha, 1[$, the relevant range for χ is $[0, 1 - \alpha[$. Now, one can easily verify that for any $(\chi, q, q') \in [0, 1 - \alpha[\times]0, 1 - \alpha] \times [0, \alpha]$, F is $\mathcal{C}^1$ and that $F'_\chi > 0$, $F'_q < 0$ and $F'_{q'} < 0$. It follows from the Implicit function theorem that (42) defines a continuous, and increasing function $\chi(q, q')$ on $]0, 1 - \alpha] \times [0, \alpha]$. This shows the first part of proposition 3. The second part follows from the characterization of $\mathbf{Q}$ (see appendix A). $\square$

C. Characterization of $\mathcal{BR}\left(\sigma^{-h}\right)$ for Γ_ε

In this section, we characterize the solution to the following optimization problem

$$\begin{aligned} \max_{\substack{0 \leq b_2, \\ 0 \leq q_1 \leq e_1^h \frac{1}{1+\varepsilon}}} \quad & \ln\left(e_1^h - (1+\varepsilon)\, q_1\right) + \beta^h \ln\left(\tfrac{b_2}{b_2 + B_2^{-I}} Q_2 + e_2^h\right) \\ \text{s.t.} \quad & b_2 \leq \tfrac{q_1}{q_1 + Q_1^{-h}} B_1. \end{aligned} \tag{$\mathcal{P}_1$}$$

This corresponds to the problem of an agent with preferences represented by $\ln x_1 + \beta^h \ln x_2$ and endowment $\left(e_1^h, e_2^h\right)$, under the assumption that he is a seller on market 1 and a buyer on market 2.

First note that the budget constraint must hold as an equality at the optimum so that the optimal bid can be written as a (concave) function $b_2(q_1) := \frac{q_1}{q_1+Q_1^{-h}} B_1$. The optimal offer thus solves

$$\max_{q_1 \in \left[0, \frac{e_1^h}{1+\varepsilon}\right]} \ln\left(e_1^h - (1+\varepsilon) q_1\right) + \beta^h \ln\left(\frac{b_2(q_1)}{b_2(q_1) + B_2^{-I}} Q_2 + e_2^h\right). \tag{43}$$

Now, program (43) is concave and admits a unique solution. This unique solution clearly satisfies $q_1 < \frac{e_1^h}{1+\varepsilon}$. An interior solution ($q_1 > 0$) then satisfies the first order condition

$$-\frac{1+\varepsilon}{e_1^h - (1+\varepsilon) q_1} + \beta^h B_1 Q_2 \frac{Q_1^{-I}}{(Q_1)^2} \frac{B_2^{-I}}{(B_2)^2} \left(\frac{b_2(q_1)}{b_2(q_1) + B_2^{-I}} Q_2 + e_2^h\right)^{-1} = 0. \tag{44}$$

The solution is $q_1 = 0$ whenever the left hand side of (44) evaluated at 0 is negative, that is iff

$$\beta^h \frac{Q_1^{-I}}{Q_2} \frac{B_2^{-I}}{B_1} \leq \frac{e_2^h}{e_1^h} (1+\varepsilon). \tag{45}$$

To summarize, there is a unique solution (q_1, b_2) to $(\mathcal{P}_1)$, characterized by $(q_1, b_2) = (0, 0)$ if (45) holds, and by $b_2 = \frac{q_1}{q_1+Q_1^{-h}} B_1$ and the FOC

$$\frac{1+\varepsilon}{e_1^h - (1+\varepsilon) q_1} = \beta^h B_1 Q_2 \frac{Q_1^{-I}}{(Q_1)^2} \frac{B_2^{-I}}{(B_2)^2} \left(\frac{b_2}{b_2 + B_2^{-I}} Q_2 + e_2^h\right)^{-1}, \tag{46}$$

otherwise. □

D. Proof of proposition 4

The proof proceeds in several steps.

Step 1. We first observe that $B_1 = B_2$ holds at a candidate active equilibrium of Γ_ε. Given lemma 2, the set $\mathcal{B}_1$ of agents for which $b_1^h > 0$ is equal to the set of agents for which $q_2^h > 0$. Now, summing over $\mathcal{B}_1$ yields $B_1 = \sum_{b_1^h>0} b_1^h = \sum_{b_1^h>0} \frac{q_2^h}{Q_2} B_2 = B_2 \left(\sum_{q_2^h>0} q_2^h\right) / Q_2 = B_2$, where the second equality follows from the budget constraint of agents in $\mathcal{B}_1$.

Step 2. We now derive a condition on (relative) prices for an agent to buying a given good. Precisely, we show:

Lemma 3. *Consider an agent h with preferences $\ln x_1 + \beta^h \ln x_2$ and endowment $\left(e_1^h, e_2^h\right)$. In equilibrium, $b_2^h > 0$ if and only if $\frac{p_2}{p_1} < \frac{1}{1+\varepsilon} \beta^h e_1^h / e_2^h$.*

Proof. ($\Rightarrow$) Assume $b_2^h > 0$. By optimality, h's strategy $\left(q_1^h, b_2^h\right)$ must satisfy $b_2^h = q_1^h p_1$ and the FOC (44) derived in appendix C. Using successively $Q_1^{-h} = Q_1 - q_1^h$ and $q_1^h = b_2^h \frac{Q_1}{B_1}$, Eq. (44) can be rearranged to

$$(1+\varepsilon)\left(b_2^h \frac{Q_2}{B_2} + e_2^h\right) = \beta^h \frac{B_1}{Q_1}\frac{Q_2}{B_2}\left(1 - \frac{b_2^h}{B_1}\right)\left(1 - \frac{b_2^h}{B_2}\right)\left(e_1^h - (1+\varepsilon)\, b_2^h \frac{Q_1}{B_1}\right),$$

and, using $B_1 = B_2$ and the definition of prices, to

$$(1+\varepsilon)\left(b_2^h + e_2^h p_2\right) = \beta^h \left(\frac{B_1 - b_2^h}{B_1}\right)^2 \left(e_1^h p_1 - (1+\varepsilon)\, b_2^h\right). \tag{47}$$

As $b_2^h > 0$, it easily follows from (47) that $(1+\varepsilon)\, e_2^h p_2 < \beta^h e_1^h p_1$. ($\Leftarrow$) Assume $(\mathcal{A}) : p_2/p_1 < \frac{1}{1+\varepsilon}\beta^h e_1^h / e_2^h$. We have to show that $b_2^h > 0$ must hold. Assume the contrary, that is $b_2^h = 0$. There are two cases, depending on whether $b_1^h > 0$ or $b_1^h = 0$. Consider first the former case. Then, by a reasoning analogous to point ($\Rightarrow$), it must hold that $(1+\varepsilon)\, e_1^h p_1 < \left(\beta^h\right)^{-1} e_2^h p_2$. But this is equivalent to $p_2/p_1 > \beta^h e_1^h / e_2^h\, (1+\varepsilon)$, which is ruled out by $(\mathcal{A})$. Hence, we are left with the case $b_1^h = 0$. In that case, for $b_2^h = 0$ to be optimal, Cond. (45) must hold. But given that $b_1^h = 0 = q_2^h$, this is equivalent to $\beta^h p_1/p_2 \leq (1+\varepsilon)\, e_2^h/e_1^h$. Using $(\mathcal{A})$, we get $\beta^h < \beta^h$, a contradiction. Hence, $b_2^h > 0$ must hold. $\square$

Step 3. We now prove the result. First, it easily follows from step 2 that prices at a candidate active equilibrium must satisfy

$$\frac{1}{1+\varepsilon}\frac{1}{\beta}\frac{\alpha}{1-\alpha} < \frac{p_2}{p_1} < \frac{1}{1+\varepsilon}\beta\frac{1-\alpha}{\alpha}. \tag{48}$$

To see this, simply note that by lemma 3 if $\frac{1}{1+\varepsilon}\frac{1}{\beta}\frac{\alpha}{1-\alpha} \leq \frac{p_2}{p_1}$ then there are no buyers on market 1. By an analogous reasoning if $\frac{p_2}{p_1} \leq \frac{1}{1+\varepsilon}\beta\frac{1-\alpha}{\alpha}$ there are no buyers on market 2. Thus (48) is the only configuration compatible with (active) equilibrium, implying in particular that $b_2^h > 0$ for all type I agents, and $b_1^h > 0$ for all type II agents. Consider first a type I agent. His bid $b_2^h > 0$ must satisfy Eq. (47), which depends only on aggregate prices (and primitives). Hence, $b_2^h = b_2^I$ for all type I agents, with

$$(1+\varepsilon)\left(b_2^I + \alpha p_2\right) = \beta\left(\frac{B_1 - b_2^h}{B_1}\right)^2 \left((1-\alpha)\, p_1 - (1+\varepsilon)\, b_2^I\right). \tag{49}$$

By a symmetric reasoning, $b_1^h = b_1^{II}$ for all type II agents, with

$$(1+\varepsilon)\left(b_1^{II} + \alpha p_1\right) = \beta\left(\frac{B_1 - b_2^h}{B_1}\right)^2 \left((1-\alpha)\, p_2 - (1+\varepsilon)\, b_1^{II}\right). \tag{50}$$

Now, $B_1 = B_2$ implies $nb_1^{II} = nb_2^I$, and therefore $b_1^{II} = b_2^I = b$. To conclude that the equilibrium is a CSNE, we need to consider offer. By the budget constraint, all type I agents offer $q_1^I = b/p_1$, and all type II agents offer $q_2^{II} = b/p_2$. Thus, we need to show that $p_1 = p_2$ in equilibrium. Substituting into (49) and (50), (b, p_1, p_2) must solve

$$\begin{cases} (1+\varepsilon)(b+\alpha p_2) = \beta \left(\frac{n-1}{n}\right)^2 ((1-\alpha) p_1 - (1+\varepsilon) b), \\ (1+\varepsilon)(b+\alpha p_1) = \beta \left(\frac{n-1}{n}\right)^2 ((1-\alpha) p_2 - (1+\varepsilon) b). \end{cases} \tag{51}$$

Subtracting the second equation from the first, we get

$$(1+\varepsilon)\alpha(p_2 - p_1) = \beta\left(\frac{n-1}{n}\right)^2 (1-\alpha)(p_1 - p_2), \tag{52}$$

implying $p_1 = p_2$. This terminates the proof. □

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